# 1: "Category Theory.zip" and Simplicial Sets

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These notes roughly follow the contents of [Wag25, §1] and [Lan21, §1]

### 1 Adjunctions and equivalences

**Definition 1.1.** Let  $L: \mathcal{C} \hookrightarrow \mathcal{D}: R$  be functors. Write  $L \dashv R$  if there exists an iso

$$\operatorname{Hom}_{\mathcal{D}}(L-,-) \cong \operatorname{Hom}_{\mathcal{C}}(-,R-)$$

in Fun( $\mathcal{C}^{op} \times \mathcal{D}$ , **Set**).

**Lemma 1.2.**  $L: C \longrightarrow D$  has a right adjoint  $\iff$  every presheaf

$$\operatorname{Hom}_{\mathcal{D}}(L-,d): \mathcal{C}^{\operatorname{op}} \longrightarrow \mathbf{Set}$$

is representable. (A dual result holds for left adjoints.)

*Proof.* ⇒ is clear, so we show  $\Leftarrow$ . Identify  $\operatorname{Hom}_{\mathcal{D}}(L-,-)$ :  $\mathcal{C}^{\operatorname{op}} \times \mathcal{D} \longrightarrow \operatorname{\mathbf{Set}}$  with  $F \colon \mathcal{D} \longrightarrow \operatorname{PSh}(\mathcal{C})$ . By assumption, all  $Fd \in \operatorname{EssIm}(\mathfrak{k})$ . Let  $\mathfrak{k}^{-1} \colon \operatorname{EssIm}(\mathfrak{k}) \longrightarrow \mathcal{C}$  be a quasi-inverse. The composite

$$\mathcal{D} \xrightarrow{F} \operatorname{EssIm}(\mathfrak{z}) \xrightarrow{\mathfrak{z}^{-1}} \mathcal{C}$$

defines a right adjoint R.

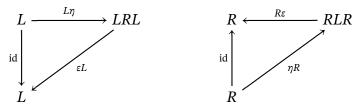
**Remark 1.3.** Let  $L: C \Longrightarrow \mathcal{D}: R$  be adjoints  $(L \dashv R)$ . The iso

$$\operatorname{Hom}_{\mathcal{D}}(Lc, Lc) \cong \operatorname{Hom}_{\mathcal{C}}(c, RLc)$$

sends  $id_{Lc}$  to  $\eta_c$ :  $c \longrightarrow RLc$ . This defines the unit  $\eta$ :  $id \Longrightarrow RL$ . The iso

$$\operatorname{Hom}_{\mathcal{C}}(Rd,Rd) \cong \operatorname{Hom}_{\mathcal{D}}(LRd,d)$$

sends  $id_{Rc}$  to  $\varepsilon_d$ :  $LRd \rightarrow d$ . This defines the counit  $\varepsilon$ :  $LR \Longrightarrow id$ . They satisfy the triangle identities.



**Exercise 1.4.** Let  $L: \mathcal{C} \leftrightarrows \mathcal{D} : R$  be functors. TFAE:

- 1.  $L \dashv R$
- 2. There exist  $\eta$ : id  $\Longrightarrow RL$  and  $\varepsilon$ :  $LR \Longrightarrow$  id that satisfy the triangle identities.

**Corollary 1.5.** Let  $L: C \Longrightarrow \mathcal{D}: R$  be adjoints,  $\mathcal{J}$  any category. Then we have adjunctions

$$L_*$$
: Fun( $\mathcal{J}, \mathcal{C}$ )  $\Longrightarrow$  Fun( $\mathcal{J}, \mathcal{D}$ ) :  $R_*$ 
 $R^*$ : Fun( $\mathcal{C}, \mathcal{J}$ )  $\leftrightarrows$  Fun( $\mathcal{D}, \mathcal{J}$ ) :  $L^*$ 

*Proof.* It suffices to construct the unit and counit. These are inherited from  $L \dashv R$ .  $\Box$  **Lemma 1.6.** Let  $L: C \leftrightarrows \mathcal{D}: R$  be an adjunction.

- 1. L is fully faithful  $\iff \eta$ : id  $\implies$  RL is an iso.
- 2. If L is fully faithful and R reflects isos, then  $L \dashv R$  is an adjoint equivalence.

*Proof.* Let  $x, y \in C$ . By definition of  $\eta$  (see Remark 1.3) we can factor  $\sharp(\eta_y)_x$  via

$$\operatorname{Hom}_{\mathcal{C}}(x,y) \xrightarrow{L} \operatorname{Hom}_{\mathcal{D}}(Lx,Ly) \xrightarrow{\operatorname{adjunction}} \operatorname{Hom}_{\mathcal{C}}(x,RLy)$$

$$\stackrel{\sharp}{\underset{\sharp(\eta_{v})_{x}}{\longrightarrow}} \operatorname{Hom}_{\mathcal{C}}(x,RLy)$$

It follows that

$$L$$
 is fully faithful  $\iff$  all  $\sharp(\eta_y)_x$  are bijections  $\iff$  all  $\sharp(\eta_y)$  are isos  $\iff$   $\eta$  is an iso.

This shows 1. To see 2. consider the triangle identity

$$R \xrightarrow{\eta R} RLR \xrightarrow{R\varepsilon} R$$

and note that if  $\eta$  is an iso, then so is  $\eta R$ . Hence  $R\varepsilon$  is an iso, and thus  $\varepsilon$  is an iso.  $\square$ 

#### 2 Limits and colimits

**Definition 2.1.** Let  $\mathcal{J}$  be a category. Denote by const:  $\mathcal{C} \longrightarrow \operatorname{Fun}(\mathcal{J}, \mathcal{C})$  the functor that sends  $c \in \mathcal{C}$  to the constant diagram.

A limit of  $F: \mathcal{J} \longrightarrow \mathcal{C}$  is an object  $\lim F \in \mathcal{C}$  satisfying

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{J},\mathcal{C})}(\operatorname{const}_{(-)},F) \cong \operatorname{Hom}_{\mathcal{C}}(-,\lim F).$$

A colimit of F is an object colim  $F \in C$  satisfying

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{J},\mathcal{C})}(F,\operatorname{const}_{(-)})\cong \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F,-).$$

**Remark 2.2.** If all diagrams  $F: \mathcal{J} \longrightarrow \mathcal{C}$  admit (co)limits, applying Lemma 1.2 yields

$$\operatornamewithlimits{colim}_{\mathcal{I}}\dashv \operatornamewithlimits{const}\dashv \varinjlim_{\mathcal{I}}$$

for free. In particular, taking (co)limits is functorial.

**Lemma 2.3.** Let  $L: C \Longrightarrow \mathcal{D}: R$  be adjoints.

- 1. L preserves colimits.
- 2. R preserves limits.

*Proof.* Suppose  $F: \mathcal{J} \longrightarrow \mathcal{C}$  has a colimit. Using  $L_* \dashv R_*$  and Lemma 1.2 we obtain

$$\begin{split} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J},\mathcal{D})}(LF,\operatorname{const}_{(-)}) &\cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J},\mathcal{C})}(F,R\operatorname{const}_{(-)}) \\ &\cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J},\mathcal{C})}(F,\operatorname{const}_{R(-)}) \\ &\cong \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F,R-) \\ &\cong \operatorname{Hom}_{\mathcal{D}}(L\operatorname{colim} F,-). \end{split}$$

The limit case is analogous.

**Lemma 2.4.** Suppose all  $\mathcal{J}$ -shaped (co)limits exist in  $\mathcal{D}$ . Then the same holds for Fun( $\mathcal{C}$ ,  $\mathcal{D}$ ) and (co)limits are computed pointwise.

*Proof.* We have a commutative diagram

$$\operatorname{Fun}(\mathcal{C},\mathcal{D}) \xrightarrow{\operatorname{const}^F} \operatorname{Fun}(\mathcal{J},\operatorname{Fun}(\mathcal{C},\mathcal{D})) \xrightarrow{\cong} \operatorname{Fun}(\mathcal{C},\operatorname{Fun}(\mathcal{J},\mathcal{D}))$$

By Corollary 1.5 we have  $\operatorname{const}^{\mathcal{C}}_* \dashv \lim_*$ . Hence,  $\operatorname{const}^F$  has a right adjoint. The pointwise condition comes from the top-right iso. The case for colimits is analogous.  $\square$ 

**Corollary 2.5.** Hom-functors preserve limits.

*Proof.* Suppose  $F: \mathcal{J} \longrightarrow \mathcal{C}$  has a limit. For  $c \in \mathcal{C}$  we have natural isos

$$\operatorname{Hom}_{\mathcal{C}}(c, \lim F) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J}, \mathcal{C})}(\operatorname{const}_{c}, F)$$

$$\cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J}, \operatorname{PSh}(\mathcal{C}))}(\sharp \operatorname{const}_{c}, \sharp F)$$

$$\cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{J}, \operatorname{PSh}(\mathcal{C}))}(\operatorname{const}_{\sharp(c)}, \sharp F)$$

$$\cong \operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}(\sharp(c), \lim(\sharp F))$$

$$\cong \lim \operatorname{Hom}_{\mathcal{C}}(c, F -).$$

where we use that **Set** is complete and limits in PSh(C) are pointwise. The  $Hom_C(-,c)$ case is an exercise.

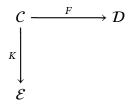
#### Kan extensions 3

Remark 3.1. We will now develop the machinery that provides the adjunctions

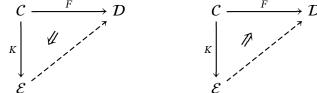
$$|\cdot|$$
: sSet  $\Longrightarrow$  Top: Sing, h: sSet  $\Longrightarrow$  Cat:N, (-) $\times$ X: sSet  $\Longrightarrow$  sSet: F(X,-)

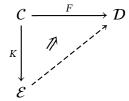
for free. To this end we have to study Kan extensions along \( \mathcal{L} \).

Suppose we are given functors F and K as follows.



Our goal is to "fill this horn" in Cat universally by a 2-cell (not necessarily as a commutative triangle) that provides a weak extension of F. Since natural transformations are oriented, there are two approaches.





**Definition 3.2.** The left Kan extension of  $F: \mathcal{C} \longrightarrow \mathcal{D}$  along  $K: \mathcal{C} \longrightarrow \mathcal{E}$  is defined via

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{E},\mathcal{D})}(F,K^*(-)) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{E},\mathcal{D})}(\operatorname{Lan}_K F,-).$$

The right Kan extension of  $F \colon \mathcal{C} \longrightarrow \mathcal{D}$  along  $K \colon \mathcal{C} \longrightarrow \mathcal{E}$  is defined via

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(K^*(-),F) \cong \operatorname{Hom}_{\operatorname{Fun}(\mathcal{E},\mathcal{D})}(-,\operatorname{Ran}_K F).$$

**Remark 3.3.** If every F admits Kan extensions, Lemma 1.2 provides adjunctions

$$\operatorname{Lan}_K(-) \dashv K^* \dashv \operatorname{Ran}_K(-)$$
.

**Exercise 3.4.** If  $\mathcal{D}$  is cocomplete, then all Lan<sub>K</sub> F exist and can be computed via

$$(\operatorname{Lan}_K F)(e) \cong \operatorname{colim}_{Kc \to e} Fc \quad (\forall e \in \mathcal{E})$$

where the colimit is over  $(K \downarrow \text{const}_e)$ .

**Corollary 3.5.** If  $\mathcal{D}$  is cocomplete and K is fully faithful, then  $\operatorname{Lan}_K(-)$  is fully faithful.

*Proof.* By Lemma 1.6 it suffices to show that each  $\eta_F$ :  $F \Longrightarrow (\operatorname{Lan}_K F) \circ K$  is an iso. Since K is fully faithful, we have  $(K \downarrow \operatorname{const}_{Kx}) \cong \mathcal{C}_{/x}$ . The latter has the terminal object  $\operatorname{id}_x$ , so taking the colimit corresponds to evaluating at this object:

$$(\operatorname{Lan}_K F)(Kx) \cong \underset{Kc \to Kx}{\operatorname{colim}} Fc \cong \underset{c \to x}{\operatorname{colim}} Fc \cong Fx$$

**Lemma 3.6.** Every presheaf is a colimit of representables. Let  $F \in PSh(C)$ . Then

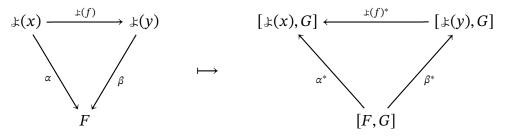
$$F \cong \underset{\sharp(c) \to F}{\operatorname{colim}} \, \sharp(c)$$

where we index over ( $\downarrow \downarrow \text{const}_F$ ).

*Proof.* For every  $G \in PSh(C)$  we have

$$\operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}\big(\operatorname{colim}_{\natural(c) \to F} \natural(c), G\big) \cong \lim_{\natural(c) \to F} \operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}\big(\natural(c), G\big) \stackrel{(\star)}{\cong} \operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}(F, G).$$

To see  $(\star)$ , note that  $\operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}(F,G)$  has a canonical cone structure by precomposition:



**Corollary 3.7.** Let  $F: \mathcal{C} \longrightarrow \mathcal{D}$ . If  $\mathcal{D}$  is cocomplete, then  $\operatorname{Lan}_{k} F$  has a right adjoint.

*Proof.* By Lemma 1.2 it suffices to show that each

$$\operatorname{Hom}_{\mathcal{D}}(\operatorname{Lan}_{\mathbb{F}} F -, d)$$
:  $\operatorname{PSh}(\mathcal{C})^{\operatorname{op}} \longrightarrow \operatorname{Set}$ 

is representable. Using the Kan extension formula, we have

$$\begin{aligned} \operatorname{Hom}_{\mathcal{D}}((\operatorname{Lan}_{\sharp}F)(X),d) &\cong \operatorname{Hom}_{\mathcal{D}}(\underset{\sharp(c) \to X}{\operatorname{colim}}Fc,d) \\ &\cong \lim_{\sharp(c) \to X} \operatorname{Hom}_{\mathcal{D}}(Fc,d) \\ &\cong \operatorname{Hom}_{\operatorname{PSh}(\mathcal{C})}(X,\operatorname{Hom}_{\mathcal{D}}(F-,d)) \end{aligned}$$

so the right adjoint is given by the functor

$$R: \mathcal{D} \longrightarrow PSh(\mathcal{C}), \quad d \longmapsto Hom_{\mathcal{D}}(F-, d).$$

**Theorem 3.8.** Suppose D is cocomplete. The Yoneda embedding  $\sharp \colon \mathcal{C} \longrightarrow \mathrm{PSh}(\mathcal{C})$  induces a weak equivalence by restriction

$$\sharp^* \colon \operatorname{Fun}^{\operatorname{colim}}(\operatorname{PSh}(\mathcal{C}),\mathcal{D}) \xrightarrow{\simeq} \operatorname{Fun}(\mathcal{C},\mathcal{D})$$

where  $\operatorname{Fun}^{\operatorname{colim}}(\operatorname{PSh}(\mathcal{C}), \mathcal{D})$  is the full subcategory of  $\operatorname{Fun}(\operatorname{PSh}(\mathcal{C}), \mathcal{D})$  spanned by the colimit preserving functors. Furthermore, every such colimit preserving functor admits a right adjoint.

*Proof.* Since each Lan<sub>k</sub> F has a right adjoint, Lan<sub>k</sub>  $\dashv k^*$  restricts to

$$\operatorname{Lan}_{\sharp} \colon \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \leftrightarrows \operatorname{Fun}^{\operatorname{colim}}(\operatorname{PSh}(\mathcal{C}), \mathcal{D}) : \sharp^*.$$

Since  $\sharp$  is fully faithful, so is  $\operatorname{Lan}_{\sharp}$  by Corollary 3.5. Finally,  $\sharp^*$  reflects isos on colimit preserving functors since every presheaf is a colimit of representables by Lemma 3.6.  $\square$ 

## 4 Simplicial sets

**Definition 4.1.** The simplex category  $\Delta$  has the following data.

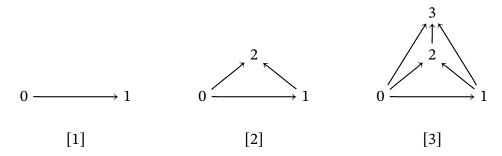
• Objects: totally ordered sets  $[n] \coloneqq \{0 < 1 < \dots < n\}$  (where  $n \in \mathbb{N}_{\geq 0}$ )

• Morphisms: monotone (i.e. weakly increasing) maps.

A simplicial set is a presheaf on  $\Delta$ , i.e. a functor  $X: \Delta^{op} \longrightarrow \mathbf{Set}$ . The category of simplicial sets is defined as  $\mathbf{sSet} := \mathrm{PSh}(\Delta)$ .

**Remark 4.2.** We can view  $\Delta$  as a full subcategory of **Cat** by identifying [n] with its posetal category.

**Remark 4.3.** The category [n] gives a combinatorial description of an n-simplex with labeled vertices.



• For each  $i \in [n]$  there is a unique injective morphism

$$d^i$$
:  $[n-1] \rightarrow [n]$ 

that does not hit *i*, called the *i*-th coface map.

• For each  $j \in [n-1]$  there is a unique surjective morphism

$$s^{j}$$
:  $[n] \longrightarrow [n-1]$ 

that hits *j* twice, called the *j*-th codegeneracy map.

For  $X \in \mathbf{sSet}$  we introduce the following notation.

- $X_n := X([n]) \in \mathbf{Set}$  are called the *n*-simplices of *X*.
- $d_i := X(d^i) \colon X_n \longrightarrow X_{n-1}$  are called the face maps of X.
- $s_j := X(s^j)$ :  $X_{n-1} \longrightarrow X_n$  are called the degeneracy maps of X.

 $x \in X_n$  is called degenerate, if  $x \in \text{im}(s_j)$  for some j.

**Definition 4.4.** The (standard) n-simplex is defined as  $\Delta^n := \sharp([n]) = \operatorname{Hom}_{\Delta}(-,[n]) \in$  **sSet**. We introduce the following sub-simplicial sets.

• 
$$\partial \Delta^n := \bigcup_{i=0}^n \operatorname{im}(\sharp(d^i): \Delta^{n-1} \longrightarrow \Delta^n)$$
, the boundary of  $\Delta^n$ ,

• 
$$\partial \Delta^n := \bigcup_{i=0}^n \operatorname{im}(\sharp(d^i) \colon \Delta^{n-1} \longrightarrow \Delta^n)$$
, the boundary of  $\Delta^n$ ,  
•  $\Lambda^n_j := \bigcup_{\substack{i=0 \ i \neq j}}^n \operatorname{im}(\sharp(d^i) \colon \Delta^{n-1} \longrightarrow \Delta^n)$ , the  $j$ -horn in  $\Delta^n$ 

Furthermore, we call  $\Lambda_j^n$  an

- inner horn, if 0 < j < n,
- outer horn, if j = 0 or j = n.

**Definition 4.5.** Let  $|\Delta^n| \in \mathbf{Top}$  be the geometric standard simplex, given by the convex hull of the standard basis vectors  $e_0, e_1, \dots, e_n \in \mathbb{R}^{n+1}$ . Given  $f : [m] \longrightarrow [n]$ , we obtain

$$|f|: |\Delta^m| \longrightarrow |\Delta^n|$$

by linearly extending  $e_i \mapsto e_{f(i)}$ . This defines a functor  $\Delta \to \mathbf{Top}$ . By Theorem 3.8 this has a unique colimit-preserving extension, the geometric realization

$$|\cdot|: \mathbf{sSet} \longrightarrow \mathbf{Top}, \quad |X| \cong \underset{\Delta^n \longrightarrow X}{\operatorname{colim}} |\Delta^n|$$

$$\Delta \longrightarrow \mathbf{Top}$$

$$|\cdot|$$
Sing

whose right adjoint must be the singular complex functor

Sing: **Top** 
$$\longrightarrow$$
 **sSet**, Sing( $X$ ) = Hom<sub>**Top**</sub>( $|\Delta^{(-)}|, X$ ).

Exercise 4.6. Show that the geometric realization of a simplicial set has a canonical CW-complex structure.

**Definition 4.7.** Every (small) category C defines a simplicial set

$$N(C): \Delta^{op} \hookrightarrow \mathbf{Cat}^{op} \xrightarrow{\mathrm{Hom}_{\mathbf{Cat}}(-,C)} \mathbf{Set}$$

called the *nerve* of C. By Theorem 3.8 we get an adjunction

$$h: \mathbf{sSet} \Longrightarrow \mathbf{Cat} : \mathbf{N}$$

where h is the unique colimit preserving extension of  $\Delta \hookrightarrow \mathbf{Cat}$ . We call h the homotopy category functor.

**Definition 4.8.** For each  $X \in \mathbf{sSet}$ , the functor  $(-) \times X$ :  $\mathbf{sSet} \longrightarrow \mathbf{sSet}$  preserves colimits (because it does so pointwise). By Theorem 3.8 it must be the extension of

$$\Delta \longrightarrow \mathbf{sSet}, \quad [n] \longmapsto \Delta^n \times X.$$

We obtain an adjunction

$$(-) \times X$$
: sSet  $\Longrightarrow$  sSet :  $F(X, -)$ 

where  $F(X,Y)_n \cong \operatorname{Hom}_{\operatorname{sSet}}(\Delta^n \times X,Y)$  defines the internal hom in **sSet**, also called the function complex.

#### References

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- [Wag25] Ferdinand Wagner. "∞-Categories in Topology". Available at https://florianadler.github.io/inftyCats/inftyCats.pdf. Sept. 2025.